



A LETTERS JOURNAL EXPLORING
THE FRONTIERS OF PHYSICS

OFFPRINT

**Analysis of parameter mismatches in the
master stability function for network
synchronization**

F. SORRENTINO and M. PORFIRI

EPL, **93** (2011) 50002

Please visit the new website
www.epljournal.org

TARGET YOUR RESEARCH WITH EPL



Sign up to receive the free EPL table of contents alert.

www.epljournal.org/alerts

Analysis of parameter mismatches in the master stability function for network synchronization

F. SORRENTINO^{1(a)} and M. PORFIRI²

¹ *Università degli Studi di Napoli Parthenope - 80143 Napoli, Italy, EU*

² *Polytechnic Institute of New York University - Brooklyn, NY 11201, USA*

received 22 December 2010; accepted in final form 7 February 2011
published online 9 March 2011

PACS 05.45.Xt – Synchronization; coupled oscillators
PACS 05.45.Pq – Numerical simulations of chaotic systems
PACS 89.75.-k – Complex systems

Abstract – In this letter, we perform a sensitivity analysis on the master stability function approach for the synchronization of networks of coupled dynamical systems. More specifically, we analyze the linear stability of a nearly synchronized solution for a network of coupled dynamical systems, for which the individual dynamics and output functions of each unit are approximately identical and the sums of the entries in the rows of the coupling matrix slightly deviate from zero. The motivation for this parametric study comes from experimental instances of synchronization in human-made or natural settings, where ideal conditions are difficult to observe.

Copyright © EPLA, 2011

Introduction. – Synchronization of networks of coupled dynamical systems has been the subject of intensive research, see for example the reviews [1–4]. Chaos synchronization of networked dynamical systems finds applications in secure communication [5–7], system identification [8–11], data assimilation [12,13], sensors [14], information encoding and transmission [15,16], and multiplexing [17]. In this framework, the master stability function analysis provides a necessary and sufficient condition for the linear stability of the synchronous solution. However, most of the research on this approach focuses on ideal conditions, which are difficult to implement in experiments.

We consider a typical experimental scenario for a set of dynamical systems that are coupled through a network to achieve synchronization. We assume that each of the elements which constitute the experiment is selected to reflect certain nominal characteristics; yet, we allow these components to be affected by small mismatches from their nominal values. We consider a wide range of possible deviations from nominal operating conditions that may affect simultaneously the individual units' dynamics, the individual units' output functions, and the coupling gains among the systems. Another motivation for the proposed analysis is the study of the collective behavior of biological groups, where individuals are generally different in nature

and their couplings are typically affected by fluctuations about an average or *nominal* value; see for example [18].

We consider the following equations of motion for a set of coupled chaotic systems in their nominal conditions:

$$\dot{x}_i(t) = F(x_i(t)) + \sigma \sum_{j=1}^N A_{ij}^{NOM} H(x_j(t)), \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i \in \mathbb{R}^n$ is the n -dimensional vector describing the state of node i , $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ governs the uncoupled dynamics of node i , $H: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vectorial output function, σ is a scalar gain describing the overall coupling strength, and N is the number of nodes in the network. The network is defined by the matrix $A^{NOM} = \{A_{ij}^{NOM}\}$, describing the coupling from node j to node i . We refer to equation set (1) as nominal, as we assume that it corresponds to a given experimental design. A sufficient condition for the existence of a synchronized solution,

$$x_1(t) = x_2(t) = \dots = x_N(t) = x_s(t), \quad (2)$$

is that

$$\sum_j A_{ij}^{NOM} = 0, \quad i = 1, \dots, N, \quad (3)$$

that is, all the row-sums¹ of the matrix A^{NOM} are equal to zero. In case condition (3) is satisfied, a synchronized

¹In what follows, we sometimes refer to the *row-sums* of a matrix, indicating with this terminology the sums of the entries along the rows of the matrix. We further comment that the analysis stays unaltered if the right-hand side of (3) equals a constant.

(a)E-mail: fsorrent@unina.it

solution $x_s(t)$ exists that satisfies

$$\dot{x}_s(t) = F(x_s(t)). \quad (4)$$

We use $\ell_1^{NOM}, \dots, \ell_N^{NOM}$ to identify the eigenvalues of the matrix A^{NOM} , which are in general complex numbers. Note that (3) implies that A^{NOM} has one eigenvalue, $\ell_1^{NOM} = 0$, with associated right eigenvector $1_N = [1, 1, \dots, 1]$.

The linear stability of (2) can be assessed by using the master stability function [19,20]. Within this framework, the synchronous solution is stable if the maximum Lyapunov exponent associated with the parametric equation

$$\dot{\gamma}(t) = [DF(x_s(t)) + cDH(x_s(t))]\gamma(t) \quad (5)$$

is negative for every $c = \sigma \ell_k^{NOM}$, $k = 2, \dots, N$, where γ is an n -dimensional vector. Then it is possible to associate a master stability function to eq. (5), which yields the maximum Lyapunov exponent of (5) as a function of the parameter c . Thus stability of the synchronized solution can be assessed for any given network described by eqs. (1) by verifying that the corresponding eigenvalues $\sigma \ell_k^{NOM}$, $k = 2, \dots, N$, are within the region of the complex plane for which the master stability function is negative.

Problem statement. – The assumptions underlying the set of equations (1) are that

i) the individual units are all described by the same dynamics $\dot{x}_i(t) = F(x_i(t))$;

ii) the systems' outputs are all described by the same function H ;

iii) the sums of the rows of the matrix A^{NOM} are all zero, that is, condition (3) is verified at each node $i = 1, \dots, N$.

While assumptions i), ii), and iii) can be easily reproduced in a numerical simulation, their practical implementation in experiments is challenging. Qualitatively good satisfaction of i), ii), and iii) in experimental instances of synchronization often requires fine tuning [21–28]. In [27–29], an adaptive strategy to dynamically preserve synchronization in the presence of slow *a priori* unknown time variations of the couplings is proposed. Though such strategy is able to preserve condition iii) in the presence of external perturbations, the row-sums of the coupling matrix are typically nonzero over the time scale of the adaptation.

In [30,31], assumption i) is removed and the effect of small mismatch of the individual units is considered. That is, these works consider the case where F in eq. (1) is replaced by F_i and the difference between F and F_i is small. In this letter, we extend the considerations of [30,31] to simultaneously allow for deviations from the exact satisfaction of all three of the assumptions i), ii), and iii). Namely, we assume that i), ii), and iii) are nominal design conditions, which might not be exactly reproduced in an experiment. We show that if

all the mismatches are small as compared to the nominal conditions, the linear stability of the nearly synchronized solution can be studied by using an extended master stability function. Moreover, when the nearly synchronous evolution is stable, the mismatches introduce forcing terms in the parametric equation that maintain the network in a state of approximate synchronization.

To take into account approximate, rather than exact satisfaction of i), ii), and iii), we rewrite the network equations in the form

$$\dot{x}_i(t) = F(x_i(t), m_i) + \sigma \sum_{j=1}^N A_{ij} H(x_j(t), p_j), \quad (6)$$

$i = 1, 2, \dots, N$, where A_{ij} represents the coupling from node j to node i , m_i is a parameter used to identify variations of the dynamics at each node i , and p_i is a parameter of the output function of each node i . We assume that $m_i = \bar{m} + \delta m_i$, where $\bar{m} = N^{-1} \sum_i m_i$ and δm_i is a small mismatch. Similarly, we write $p_i = \bar{p} + \delta p_i$, where $\bar{p} = N^{-1} \sum_i p_i$ and δp_i is a small mismatch. Note that by construction $\sum_i \delta m_i = 0$ and $\sum_i \delta p_i = 0$. The elements A_{ij} 's represent imperfect realizations of the nominal couplings A_{ij}^{NOM} 's, that is, $A_{ij} = A_{ij}^{NOM} + \delta A_{ij}$, $i, j = 1, \dots, N$, where δA_{ij} is a small mismatch. In general, in the presence of deviations of the A_{ij} 's from their nominal values, it is not possible to write a condition equivalent to (3) and thus to extend directly the master stability function formalism. For small δA_{ij} 's, we can write

$$\sum_j A_{ij} = \sum_j \delta A_{ij} = \delta \bar{a} + \delta a_i, \quad (7)$$

where

$$\delta \bar{a} = N^{-1} \sum_{i,j} A_{ij} = N^{-1} \sum_{i,j} \delta A_{ij} \quad (8)$$

is the average sum of the rows of the matrix A and

$$\delta a_i = \left(\sum_j \delta A_{ij} \right) - \delta \bar{a} = \left(\sum_j \delta A_{ij} \right) - N^{-1} \sum_{i,j} \delta A_{ij} \quad (9)$$

is a small deviation. The deviations δa_i are calculated with respect to the average row-sum $\delta \bar{a}$, hence they have zero sum, that is, $\sum_j \delta a_j = 0$. By using condition (7) in equation set (6), we obtain

$$\begin{aligned} \dot{x}_i(t) = & F(x_i(t), m_i) + \sigma \sum_j A'_{ij} H(x_j(t), p_j) \\ & + \sigma \delta a_i H(x_i(t), p_i), \end{aligned} \quad (10)$$

$i = 1, 2, \dots, N$, where we have introduced the matrix A' defined by

$$A'_{ij} = \begin{cases} A_{ij}, & \text{if } j \neq i, \\ A_{ii} - \delta a_i, & \text{if } j = i. \end{cases} \quad (11)$$

By construction, the matrix $A' = \{A'_{ij}\}$ is such that the sums of its rows are constant and equal to $\delta \bar{a}$. We note

that by setting to zero all the mismatches δm_i , δp_i , and δa_i in (10), a synchronized solution exists for the set of equations in (10) of the form

$$\dot{\tilde{x}}_s = F(\tilde{x}_s, \tilde{m}) + \sigma \delta \bar{a} H(\tilde{x}_s, \tilde{p}). \quad (12)$$

Extended master stability function. – We introduce the average trajectory $\bar{x}(t) = N^{-1} \sum_k x_k(t)$ that satisfies the following average dynamics:

$$\begin{aligned} \dot{\bar{x}}(t) = N^{-1} & \left[\sum_k F(x_k(t), m_k) + \sigma \sum_{k,j} A'_{kj} H(x_j(t), p_j) \right. \\ & \left. + \sigma \sum_k \delta a_k H(x_k(t), p_k) \right]. \end{aligned} \quad (13)$$

Since the quantities δa_i , δp_i , and δm_i are small, we expect the individual trajectories $x_i(t)$ to be *close* to the average trajectory $\bar{x}(t)$, that is, $\|x_i(t) - \bar{x}(t)\| \leq k^*$ for all times and some small $k^* > 0$. We define the variation with respect to the average trajectories as $\delta x_i(t) = (x_i(t) - \bar{x}(t))$. By expanding both (6) and (13) to first order about $(\bar{x}(t), \tilde{m}, \tilde{p})$, we obtain

$$\begin{aligned} \delta \dot{x}_i(t) = & DF_x(\bar{x}(t), \tilde{m}) \delta x_i(t) + DF_m(\bar{x}(t), \tilde{m}) \delta m_i \\ & + \sigma DH_x(\bar{x}(t), \tilde{p}) \sum_j (A'_{ij} - b_j) \delta x_j(t) \\ & + \sigma DH_p(\bar{x}(t), \tilde{p}) \sum_j (A'_{ij} - b_j) \delta p_j \\ & + \sigma H(\bar{x}(t), \tilde{p}) \delta a_i, \end{aligned} \quad (14)$$

$i = 1, \dots, N$, where $b_j = N^{-1} \sum_k A'_{kj}$, that is, b_j represents the sum of the entries over column j of the matrix A' divided by N , for $j = 1, \dots, N$. We have indicated with DF_x and DH_x the partial derivative of the functions F and H with respect to x , with DF_m the partial derivative of the function F with respect to m , and with DH_p the partial derivative of the function H with respect to p . To obtain (14), we have used the properties $\sum_j \delta x_j = 0$, $\sum_j \delta a_j = 0$, $\sum_j \delta m_j = 0$, $\sum_j \delta p_j = 0$, $\sum_j b_j = \delta \bar{a}$, and we have discarded second-order terms in all the variations.

We define $\ell'_1, \ell'_2, \dots, \ell'_N$ as the eigenvalues of the matrix A' . We note that since the row-sums of the matrix A' are equal to $\delta \bar{a}$, the matrix A' has one eigenvalue $\ell'_1 = \delta \bar{a}$, with associated right eigenvector 1_N . Now, we consider the matrix $\tilde{A} = \{\tilde{A}_{ij}\}$, where $\tilde{A}_{ij} = (A'_{ij} - b_j)$, and we look for the solutions of the eigenvalue equation $\tilde{A} \tilde{v}_i = \lambda_i \tilde{v}_i$. We observe that the matrix \tilde{A} has the property that both the sums of its rows and its columns are equal zero. Thus $\tilde{v}_1 = 1_N$ is still a right eigenvector for the matrix \tilde{A} , with associated eigenvalue $\lambda_1 = 0$. Moreover, $\tilde{w}_1 = 1_N$ is also the left eigenvector of the matrix \tilde{A} , associated with the eigenvalue 0. The remaining eigenvalues of the matrix \tilde{A} are $\lambda_i = \ell'_i$ for $i = 2, \dots, N$ [31]. In other words, the matrices A' and \tilde{A} have the same spectrum except for the eigenvalue associated with the right eigenvector $\tilde{v}_1 = 1_N$. As discussed in what follows, the eigenvalues $\lambda_2, \dots, \lambda_N$ control the stability of the nearly synchronous solution.

Equations (14) can be rewritten as

$$\begin{aligned} \delta \dot{X}(t) = & [I_N \otimes DF_x(\bar{x}(t), \tilde{m}) + \sigma \tilde{A} \otimes DH_x(\bar{x}(t), \tilde{p})] \delta X(t) \\ & + [I_N \otimes DF_m(\bar{x}(t), \tilde{m})] \delta M \\ & + \sigma [\tilde{A} \otimes DH_p(\bar{x}(t), \tilde{p})] \delta P \\ & + \sigma [I_N \otimes H(\bar{x}(t), \tilde{p})] \delta A, \end{aligned} \quad (15)$$

where $\delta X(t) = [\delta x_1(t)^T, \delta x_2(t)^T, \dots, \delta x_N(t)^T]^T$, $\delta M = [\delta m_1, \delta m_2, \dots, \delta m_N]^T$, $\delta P = [\delta p_1, \delta p_2, \dots, \delta p_N]^T$, $\delta A = [\delta a_1, \delta a_2, \dots, \delta a_N]^T$, and the symbol \otimes indicates direct product or Kronecker product.

Following [20] and assuming that the matrix \tilde{A} is diagonalizable, we write, $\tilde{A} = V \Lambda W$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$, V is a matrix whose columns are the right eigenvectors of the matrix \tilde{A} , and $W = V^{-1}$. Premultiplying (15) by $W \otimes I_n$, we obtain

$$\begin{aligned} \dot{Q}(t) = & [I_N \otimes DF_x(\bar{x}(t), \tilde{m}) + \sigma \Lambda \otimes DH_x(\bar{x}(t), \tilde{p})] Q(t) \\ & + [W \otimes DF_m(\bar{x}(t), \tilde{m})] \delta M \\ & + \sigma [\lambda_i W \otimes DH_p(\bar{x}(t), \tilde{p})] \delta P \\ & + \sigma [W \otimes H(\bar{x}(t), \tilde{p})] \delta A, \end{aligned} \quad (16)$$

where $Q(t) = (W \otimes I_n) \delta X(t)$. We note that both matrices I_N and Λ in the homogeneous part of eq. (16) are diagonal matrices. Thus eq. (16) can be decomposed into N blocks of the form

$$\begin{aligned} \dot{q}_i(t) = & [DF_x(\bar{x}(t), \tilde{m}) + \sigma \lambda_i DH_x(\bar{x}(t), \tilde{p})] q_i(t) \\ & + \sum_j W_{ij} \delta m_j DF_m(\bar{x}(t), \tilde{m}) \\ & + \sigma \lambda_i \sum_j W_{ij} \delta p_j DH_p(\bar{x}(t), \tilde{p}) \\ & + \sigma \sum_j W_{ij} \delta a_j H(\bar{x}(t)), \end{aligned} \quad (17)$$

$i = 1, \dots, N$. We comment that the homogeneous part of each block in (17) is independent of the other blocks. For $i = 1$, the variational equation (17) yields $q_1(t) = 0$ since $\sum_{i=1}^N \delta x_i(t) = 0$ and 1_N is a left eigenvector. Thus we note that the component of the evolution along the direction $x_1 = x_2 = \dots = x_N$ is not affected by the mismatches δm_i , δp_i , and δa_i . Stability of the nearly synchronized solution is controlled by perturbations in the remaining directions, q_2, \dots, q_N . Following [20,31], it is possible to associate the following parametric equation to eq. (17):

$$\begin{aligned} \dot{z}(t) = & [DF_x(\bar{x}(t), \tilde{m}) + \omega DH_x(\bar{x}(t), \tilde{p})] z(t) \\ & + \epsilon DF_m(\bar{x}(t), \tilde{m}) + \zeta DH_p(\bar{x}(t), \tilde{p}) + \eta H(\bar{x}(t)), \end{aligned} \quad (18)$$

which corresponds to equation set (17) upon setting $z = q_i$, $\omega = \sigma \lambda_i$, $\epsilon = \sum_j W_{ij} \delta m_j$, $\zeta = \sigma \lambda_i \sum_j W_{ij} \delta p_j$, and $\eta = \sigma \sum_j W_{ij} \delta a_j$, for $i = 2, \dots, N$.

In order to assess the linear stability of the nearly-synchronous solution, eq. (18) needs to be tested for the

set of eigenvalues $\lambda_2, \dots, \lambda_N$. If the Lyapunov exponents associated with the homogeneous part of eq. (18), $i = 2, \dots, N$, are negative, the nearly synchronous solution is stable. In this case, the forcing terms on the right-hand side of eq. (18), $i = 2, \dots, N$, can be considered as inputs to a stable system. It is then possible to associate an extended master stability function $\mathcal{M}(\omega, \epsilon, \zeta, \eta)$, defined as $\lim_{\tau \rightarrow \infty} \sqrt{\tau^{-1} \int_0^\tau \|z(t)\|^2 dt}$ to eq. (18), which yields the asymptotic norm of the time average of z as a function of the tuple $(\omega, \epsilon, \zeta, \eta)$. However, stability of the nearly synchronous solution depends on the homogeneous part of (18), that is, it depends on ω , while it is independent of ϵ, ζ , and η . We note that for $\delta a_i = 0$, $\delta m_i = 0$, and $\delta p_i = 0$ with $i = 1, \dots, N$, the parametric equation (18) reduces to (5), which corresponds to the ideal case where all the parameters are equal to their nominal values.

Moreover, following [31], in the case that the master stability function is asymptotically bounded and ω is fixed, we have that $\mathcal{M}(\omega, \epsilon, \zeta, \eta)$ scales linearly with respect to ϵ, ζ , and η , that is

$$\mathcal{M}(\omega, \epsilon, \zeta, \eta) \simeq c_\epsilon(\omega)|\epsilon| + c_\zeta(\omega)|\zeta| + c_\eta(\omega)|\eta|, \quad (19)$$

where the coefficients c_ϵ, c_η , and c_ζ are functions of ω .

We comment that the extended master stability function depends on the eigenvalues of the *perturbed* matrix A' and not on those of the *nominal* matrix A^{NOM} . The matrix A' can be considered a perturbed version of the nominal matrix A^{NOM} , $A' = A^{NOM} + \Delta$, where the perturbation matrix $\Delta = \{\delta A_{ij}\} = \{\delta A_{ij} - \delta^{ij}(\sum_j \delta A_{ij} - a)\}$ and δ^{ij} indicates the Kronecker delta, $i, j = 1, \dots, N$. Note the sums of the rows of Δ are equal to $\delta \bar{a}$. The eigenvalues of the perturbed matrix A' can be computed from the spectral properties of A^{NOM} . By using classical perturbation theory [32] and assuming that the eigenvalues of the matrix A^{NOM} are all distinct, we find

$$\lambda_i \simeq \ell_i^{NOM} + \frac{\hat{w}_i^T \Delta \hat{v}_i}{\hat{w}_i^T \hat{v}_i}, \quad i = 2, \dots, N, \quad (20)$$

where \hat{w}_i and \hat{v}_i are the left and right eigenvectors associated with the eigenvalues ℓ_i^{NOM} of the matrix A^{NOM} , respectively. Equation (20) shows that the deviations of the relevant eigenvalues from their nominal values are of the same order of the perturbations Δ_{ij} on the couplings. We also comment that eq. (20) predicts that $\ell'_1 \simeq (\hat{w}_1^T \Delta \mathbf{1}_N) / (\hat{w}_1^T \mathbf{1}_N) = a$, since $\ell'_1 = a$ by construction. Similar arguments can be used to estimate the left eigenvectors of \tilde{A} from the spectral properties of A^{NOM} .

The main result of our analysis is that stability of the nearly synchronous evolution for the system (6) can be assessed by using a master stability function, which depends on the eigenvalues of an appropriately modified coupling matrix A' . Though in a practical situation it is not feasible to exactly calculate these eigenvalues, for small deviations of the couplings from their nominal values they differ from their nominal values ℓ_i^{NOM} by a small quantity

of the same order of the Δ . Moreover, the mismatches in the individual functions F and H , along with the deviations in the row-sums of the coupling matrix A , introduce forcing terms in eq. (18) through the coefficients ϵ, ζ , and η . Such forcing terms maintain the network in a state of approximate synchronization.

Following [31], in case the matrix \tilde{A} has an orthonormal basis of eigenvectors, that is, it is symmetric, we can write

$$E \equiv \lim_{\tau \rightarrow \infty} \tau^{-1} \int_0^\tau \sum_i^N \|\delta x_i(t)\|^2 dt = \sum_{i=2}^N \mathcal{M}^2(\omega_i, \epsilon_i, \zeta_i, \eta_i). \quad (21)$$

We note that E is a quantity of physical interest, as it represents the time average sum, over all the coupled systems of the *distances* $\|\delta x_i(t)\|$ from the average trajectory $\bar{x}(t)$. One of the advantages of this approach is that, by computing the master stability function once, E can be estimated *for any network topology* that approximately satisfies the constant-row-sum condition.

As pointed out in [31], a complication with this approach is that eq. (18) depends on $\bar{x}(t)$, which is an averaged trajectory over all the systems in the network. In a large network, calculating $\bar{x}(t)$ may be computationally expensive, as it requires full integration of N individual systems, see eq. (13). However, for practical purposes, $\bar{x}(t)$ in (18) can be replaced by the individual dynamics $\tilde{x}_s(t)$ in (12), which depends explicitly on \bar{m}, \bar{p} , and $\delta \bar{a}$. We comment that, unless precise knowledge of the characteristics of all the individual units and of their couplings is available, it is difficult to exactly compute \bar{m}, \bar{p} , and $\delta \bar{a}$. Nevertheless, *a priori* knowledge of the statistical properties of the coupled systems can be used to infer the average parameters. For example, if m_i, p_i and a_i , with $i = 1, \dots, N$ are taken as independent and identically distributed random variables, drawn from distributions having mean corresponding to their nominal values, and finite variance, the central-limit theorem states that \bar{m}, \bar{p} , and $\delta \bar{a}$ approach their nominal values as the number of nodes increases.

Numerical simulation. – We use the algorithm in [33] to generate a scale-free network of $N = 100$ nodes with average degree equal to 30 and exponent of the power law degree distribution equal to 3. For each pair of nodes $i, j = 1, \dots, N, j \neq i$, $A_{ij}^{NOM} = A_{ji}^{NOM} = 1$ if nodes i and j are connected; otherwise, $A_{ij}^{NOM} = A_{ji}^{NOM} = 0$. We set $A_{ii}^{NOM} = -\sum_j A_{ij}^{NOM}$, which guarantees that the row-sums of the matrix A^{NOM} are equal to zero. Moreover, as the matrix A^{NOM} is symmetric, it is diagonalizable, its eigenvalues are real, and the eigenvectors can be taken to be orthonormal. We find that $\ell_2 = 12.7018$ and $\ell_N = 86.0531$, where we set $\ell_1 \leq \ell_2 \leq \dots \leq \ell_N$.

We consider that $A_{ij} = A_{ij}^{NOM}(1 + \varsigma_a \rho_{ij})$, for $i, j = 1, \dots, N$, where $\rho_{ij} = \rho_{ji}$ is a random number drawn from a standard normal distribution and ς_a is a scalar. Upon this selection, the matrix A' in (11) is symmetric; the matrix \tilde{A} is also symmetric, since

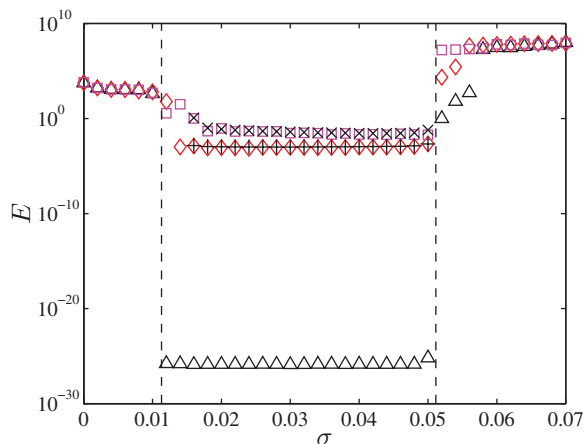


Fig. 1: (Colour on-line) Triangles, diamonds, and squares represent the error measure E vs. the coupling strength σ . Triangles are used for the case in which $\varsigma_a = \varsigma_m = \varsigma_p = 0$. Diamonds are used for the case in which $\varsigma_a = 10^{-4}$ and $\varsigma_m = \varsigma_p = 0$. Squares are used for the case in which $\varsigma_a = 10^{-4}$ and $\varsigma_m = \varsigma_p = 5 \times 10^{-4}$. The vertical dashed lines delimit the range of stability predicted by the master stability function. The symbols \times ($+$) refer to $\sum_{i=2}^N \mathcal{M}(\omega_i, \epsilon_i, \zeta_i, \eta_i)^2$ for $\varsigma_a = 10^{-4}$ and $\varsigma_m = \varsigma_p = 0$ ($\varsigma_a = 10^{-4}$ and $\varsigma_m = \varsigma_p = 5 \times 10^{-4}$), computed using eq. (19).

$b_j = \delta \bar{a} / N$ for $j = 1, \dots, N$. For $\varsigma_a = 10^{-4}$, we obtain $\lambda_2 = 12.7007$ and $\lambda_N = 86.0529$. This is in agreement with eq. (20), as we find that $|\lambda_i - \ell_i|$ is on average of the same order of magnitude of the deviations on the couplings².

We perform a numerical experiment for a set of nominally identical Rössler oscillators that are affected by mismatches in both their dynamics and output functions and are coupled by the scale free network, described by the matrix A . In this case, the equations of motion are

$$\begin{aligned} \dot{x}_{i1}(t) &= -x_{i2}(t) - x_{i3}(t) + \sigma \sum_j A_{ij}(x_{j1}(t) + p_j), \\ \dot{x}_{i2}(t) &= x_{i1}(t) + m_i x_{i2}(t), \\ \dot{x}_{i3}(t) &= 0.2 + (x_{i1}(t) - 7)x_{i3}(t), \end{aligned} \quad (22)$$

$i = 1, \dots, N$, where the state vector of oscillator i is $x_i = [x_{i1}, x_{i2}, x_{i3}]^T$. The parameters p_j are random numbers drawn from a Gaussian distribution with mean equal to zero and standard deviation ς_p , and the parameters m_i are random numbers drawn from a Gaussian distribution with mean value equal to 0.2 and standard deviation ς_m .

In fig. 1, we plot the error measure E , defined in (21), vs. the coupling strength σ . Simulations are run for a total time duration $T = 3000$, which is considerably larger than the typical time scale of an oscillation for an uncoupled Rössler oscillators, that is 2π ; time averages are taken over the time interval $[2700, 3000]$.

From the direct numerical integration of eqs. (12) and (18) with $\bar{m} = 0.2$, $\bar{p} = 0$, $\delta \bar{a} = 0$, and $\epsilon = \zeta = \eta = 0$, we find

²We have also performed numerical experiments for $\rho_{ij} \neq \rho_{ji}$ and we have found that, for small values of ς_a , the eigenvalues λ_i 's are still real.

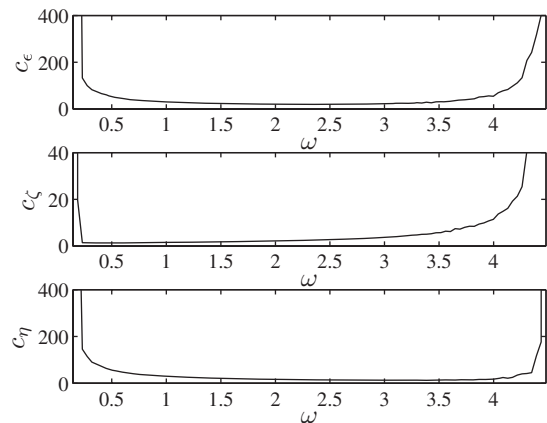


Fig. 2: c_ϵ , c_ζ , and c_η vs. ω .

that the master stability function converges to zero in the range $0.143 \lesssim \omega \lesssim 4.40$, which for our choice of the matrix A , corresponds to stability in the range $0.0113 \lesssim \sigma \lesssim 0.0511$. This range is delimited by the vertical dashed lines in fig. 1, which shows good agreement with our computations of the full nonlinear system (22). Figure 1 illustrates that the range of stability is affected neither by the presence of small deviations from the nominal couplings nor from small mismatches in the individual oscillators' parameters. This is because the eigenvalues λ_i are indistinguishable from the eigenvalues ℓ_i , for $i = 2$ or N to the degree of accuracy of the simulation shown in the figure. However, for σ inside the range of stability, the value attained by E depends on the values of δa_i , δm_i , and δp_i . Figure 2 shows c_ϵ , c_ζ , and c_η vs. ω . With this information, eq. (19) provides an estimate of the master stability function for any tuple $(\omega, \epsilon, \zeta, \eta)$. We use eq. (19) along with the data plotted in fig. 2 to calculate the master stability function \mathcal{M} . This is shown for comparison in fig. 1, where the symbols \times ($+$) are used to plot $\sum_{i=2}^N \mathcal{M}(\omega_i, \epsilon_i, \zeta_i, \eta_i)^2$ for $\varsigma_a = 10^{-4}$ and $\varsigma_m = \varsigma_p = 0$ ($\varsigma_a = 10^{-4}$ and $\varsigma_m = \varsigma_p = 5 \times 10^{-4}$). Poorer agreement is observed for values of σ slightly above the lower threshold for stability of 0.0113 (not shown), which corresponds to a so-called *bubbling* region, as further discusses below³.

Conclusions. – The master stability function analysis [19,20] provides a necessary and sufficient condition for linear stability of the synchronous solution for an arbitrary network of coupled identical systems. An extension of this approach for networks of groups, where the dynamics of nodes within a group are the same but are different for nodes in distinct groups, is proposed in [34]. In addition, a master stability function for networks in which each unit independently implements an adaptive strategy to maintain synchronization is presented in [35]. The analysis of

³Numerical experiments performed by replacing δm_i , δp_j , and δA_{ij} 's with random numbers from the same distributions and $\lambda_1, \dots, \lambda_N$ and W with the eigenvalues and eigenvectors of the original matrix A^{NOM} show good agreement with the results in fig. 1.

nearly identical coupled dynamical systems is considered in [30,31]. For this case, which is of practical relevance in experimental instances of synchronization and in biological systems, it is shown that a master stability function approach is applicable [31].

In this letter, we have proposed a sensitivity analysis to address synchronization in the presence of a broad range of deviations from nominal conditions. In particular, we have taken into consideration simultaneous small deviations in the dynamics of individual units, the output functions of the individual units, and the coupling among the systems. We have shown that the master stability function formalism can be extended to this general scenario and that stability of the nearly synchronous evolution depends on the eigenvalues of an appropriately modified coupling matrix. Our analysis is motivated by inherent practical challenges in implementing ideal conditions in experimental analysis of synchronization. For example, our approach can be directly applied to synchronization of nearly identical units whose interconnections yield to approximately zero-row-sum coupling matrix. In this case, the proposed master stability function can be used to estimate the conditions under which the nearly synchronous evolution is stable and in case of stability, the approach can be used to quantify the overall synchronization error.

Noise or small mismatches in the parameters of the individual systems can be responsible for the onset of *bubbling* [30,35,36], that is, rare intermittent large deviations from synchronization. We expect bubbling also to arise in the case of approximate satisfaction of the zero-row-sum condition; in this case, the master stability function, introduced in this letter, can be used to identify stable, unstable, and bubbling regions in the relevant parameter space, see for example [35].

FS would like to thank ED OTT for insightful discussions. MP was supported by the National Science Foundation under Grant No. CMMI-0745753.

REFERENCES

- [1] BOCCALETTI S., KURTHS J., OSIPOV G., VALLADARES D. L. and ZHOU C., *Phys. Rep.*, **366** (2002) 1.
- [2] PIKOVSKY A., ROSENBLUM M. and KURTHS J., *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press) 2003.
- [3] ARENAS A., DIAZ-GUILERA A., KURTHS J., MORENO Y. and ZHOU C., *Phys. Rep.*, **469** (2008) 93.
- [4] GONZALEZ-MIRANDA J. M., *Synchronization and Control of Chaos* (Imperial College Press, London) 2004.
- [5] CUOMO K. M. and OPPENHEIM A. V., *Phys. Rev. Lett.*, **71** (1993) 177.
- [6] ARGYRIS A., HAMACHER M., CHLOUVERAKIS K. E., BOGRIS A. and SYVRIDIS D., *Phys. Rev. Lett.*, **100** (2008) 194101.
- [7] FEKI M., *Chaos, Solitons Fractals*, **18** (2003) 141.
- [8] ABARBANEL H. D. I., CREVELING D. R. and JEANNE J. M., *Phys. Rev. E*, **77** (2008) 016208.
- [9] CREVELING D. R., GILL P. E. and ABARBANEL H. D. I., *Phys. Lett. A*, **372** (2008) 2640.
- [10] QUINN J. C., BRYANT P. H., CREVELING D. R., KLEIN S. R. and ABARBANEL H. D. I., *Phys. Rev. E*, **80** (2009) 016201.
- [11] SORRENTINO F. and OTT E., *Chaos*, **19** (2009a) 033108.
- [12] SO P., OTT E. and DAYAWANSA W. P., *Phys. Rev. E*, **49** (1994) 2650.
- [13] DUANE G. S., TRIBBIA J. J. and WEISS J. B., *Nonlinear Processes Geophys.*, **13** (2006) 601.
- [14] SORRENTINO F. and OTT E., *Phys. Rev. E*, **79** (2009b) 016201.
- [15] HAYES S., GREBOGI C. and OTT E., *Phys. Rev. Lett.*, **70** (1993) 3031.
- [16] DRONOV V., HENDREY M., ANTONSEN T. M. and OTT E., *Chaos*, **14** (2004) 30.
- [17] TSMIRING L. S. and SUSHCHIK M. M., *Phys. Lett. A*, **213** (1996) 155.
- [18] SUMPTER D. J. T., *Collective Animal Behavior* (Princeton University Press) 2009.
- [19] FUJISAKA H. and YAMADA T., *Prog. Theor. Phys.*, **69** (1983) 32.
- [20] PECORA L. and CARROLL T., *Phys. Rev. Lett.*, **80** (1998) 2109.
- [21] FINK K. S., JOHNSON G., CARROLL T. L., MAR D. and PECORA L. M., *Phys. Rev. E*, **61** (2000) 5080.
- [22] PORFIRI M. and FIORILLI F., *Physica D*, **239** (2010) 454.
- [23] PORFIRI M. and FIORILLI F., *Chaos, Solitons Fractals*, **41** (2009) 245.
- [24] POSADAS-CASTILLO C., CRUZ-HERNANDEZ C. and LOPEZ-GUTIERREZ R., *Chaos, Solitons Fractals*, **40** (2009) 1963.
- [25] BUSCARINO A., FORTUNA L. and FRASCA M., *Physica D*, **238** (2009) 1917.
- [26] SHI Z., HONG S. and CHEN K., *Phys. Lett. A*, **372** (2008) 5575.
- [27] RAVOORI B., COHEN A. B., SETTY A. V., SORRENTINO F., MURPHY T. E., OTT E. and ROY R., *Phys. Rev. E*, **80** (2009) 056205.
- [28] COHEN A. B., RAVOORI B., SORRENTINO F., MURPHY T. E., OTT E. and ROY R., unpublished (2010).
- [29] SORRENTINO F. and OTT E., *Phys. Rev. Lett.*, **100** (2008) 114101.
- [30] RESTREPO J. G., OTT E. and HUNT B. R., *Phys. Rev. E*, **69** (2004) 066215.
- [31] SUN J., BOLLT E. M. and NISHIKAWA T., *EPL*, **85** (2009) 60011.
- [32] GOLUB G. H. and LOAN C. F. V., *Matrix Computations* (The Johns Hopkins University Press) 1996.
- [33] GOH K.-I., KAHNG B. and KIM D., *Phys. Rev. Lett.*, **87** (2001) 278701.
- [34] SORRENTINO F. and OTT E., *Phys. Rev. E*, **76** (2007) 056114.
- [35] SORRENTINO F., BARLEV G., COHEN A. B. and OTT E., *Chaos*, **20** (2010) 013103.
- [36] ASHWIN P., BUESCU J. and STEWART I. N., *Phys. Lett. A*, **193** (1994) 126.