Local adaptive strategy for synchronization of complex networks

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Abstract—This paper is concerned with the analysis of the synchronization of networks of nonlinear oscillators through an innovative adaptive strategy. In particular, time-varying feedback coupling gains are considered, whose derivative is made directly dependent on the local synchronization error at each vertex in the network. It is shown that, under appropriate conditions, the strategy is indeed successful in guaranteeing the achievement of a common synchronous evolution for all oscillators in the network. The theoretical derivation is complemented by its validation on a set of representative examples.

I. INTRODUCTION

Networked systems abound in Nature and in Applied Science. Networks of dynamical systems have been recently proposed as models in many diverse fields of applications (see for instance [1], [2] and references herein).

Recently, particular attention has been focused on the problem of making a network of dynamical systems synchronize onto a common evolution. Typically, the network consists of $N$ identical nonlinear dynamical systems coupled through the edges of the network itself [1], [2]. Each uncoupled system is described by a set of nonlinear ordinary differential equations (ODEs) of the form $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a sufficiently smooth nonlinear vector field describing the system dynamics. Because of the coupling with the neighboring nodes in the network, the dynamics of each oscillator is affected by a nonlinear input representing the interaction of all neighboring nodes with the oscillator itself. Hence, the equations of motion for the generic $i$-th system in the network become:

$$\frac{dx_i}{dt} = f(x_i(t)) - \sigma \sum_{j=1}^{N} L_{ij} h(x_j), \quad i = 1, 2, \ldots, N, \quad (1)$$

where $x_i$ represents the state vector of the $i$-th oscillator, $\sigma$ the overall strength of the coupling, $h$ the output function through which the systems in the network are coupled and $L_{ij}$ the elements of the Laplacian matrix $L$ describing the network topology. In particular, $L$ is such that its entries, $L_{ij}$, are zero if node $i$ is not connected to node $j \neq i$, while are negative if node $i$ is connected to node $j$, with $|L_{ij}|$ giving a measure of the strength of the interaction.

Now, the synchronization problem is to find $\sigma$ so that all the systems in the network synchronize on the same unknown evolution, say $x_s(t)$. Specifically, one wants to find an appropriate value of $\sigma$ such that

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0, \quad (2)$$

for all pairs $(i, j)$ of nodes such that $i \neq j$. An approach based on the so-called Master Stability Function (MSF) has been established to evaluate the range of $\sigma$ for which synchronization is attained [1]. Basically, it has been shown that the width of the range of values of the coupling gain associated with the synchronous behavior is related to the ratio between the largest and smallest non-zero eigenvalue of the network Laplacian $L$. Such a ratio is often referred to as the network eigenratio. The MSF gives an indication of the network synchronizability, in terms of the values of $\sigma$ for which synchronization can be achieved.

In general, the coupling gain is chosen to be identical for all edges in the network and constant in time. Many real-world networks are characterized instead by evolving, adapting coupling gains, which vary in time according to different environmental conditions. A typical example are wireless sensor networks that gather and communicate data to a central base station [3]. Adaptation is also necessary to control networks of robots when the conditions change unexpectedly (i.e. a robot loses a sensor) [4]. Moreover, examples of adaptive networks can be found in biology, as the social insect colonies, described in [5].

In their recent work [6], Chen and Liu have considered a unique adaptive coupling gain, equal for each node of the network, chosen on the basis of global information. In the networks described above, it is more likely that the choice is local so that the coupling gain is different for each node. In [7], a new local strategy, named vertex-based strategy, has been introduced. An alternative strategy was also independently described in [8]. It was shown numerically that it is a viable approach to synchronization, but the asymptotic stability is not analytically proved. The aim of this paper is to give sufficient conditions for the global asymptotic stability of the network under this strategy.
II. MODEL DESCRIPTION AND PRELIMINARIES

A. Model description

Throughout the paper, we will always refer to the following network model:

$$ \frac{dx_i}{dt} = f(x_i) - \sigma_i(t) \sum_{j=1}^{N} C_{ij} h(x_j), \quad i = 1, 2, \ldots, N \quad (3) $$

with \( \sigma_i \) being the adaptive gain associated to the \( i \)-th node.

The vertex-based strategy in [7], where each node has a different coupling gain, is described by the following adaptation law:

$$ \dot{\sigma}_i(t) = \mu \left\| \sum_{j \in V_i} \left( h(x_j) - h(x_i) \right) \right\|, \quad i = 1, 2, \ldots, N \quad (4) $$

where \( V_i \) is the set of neighbours of the node \( i \).

B. Preliminaries

In this subsection we introduce the notations and some definitions that will be used throughout the rest of the paper.

Definition 1. As in [6], we say that a function \( f : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}^n \) is QUAD iff, for any \( x, y \in \mathbb{R}^n \):

$$ (x - y)^T f(x, t) - f(y, t) - (x - y)^T \Delta (x - y) \leq -\varpi \cdot (x - y), \quad (5) $$

where \( \Delta \) is an arbitrary diagonal matrix of order \( n \) and \( \varpi \) is a positive scalar.

Definition 2. If \( M \) is a \( q \)-dimensional square matrix and \( v \) is a \( q \)-dimensional vector, we define:

$$ |Mv| = \begin{bmatrix} |(Mv)_1| \\ \vdots \\ |(Mv)_q| \end{bmatrix} \quad (6) $$

and

$$ v \odot M = \begin{bmatrix} v_1 M_{11}, & \cdots, & v_1 M_{1q} \\ \cdots & \ddots & \cdots \\ v_q M_{q1}, & \cdots, & v_q M_{qq} \end{bmatrix}. \quad (7) $$

In what follows we define \( \Xi = \text{diag}\{\xi_1, \ldots, \xi_n\} \) and \( U = \Xi - \xi \xi^T \), where \( \xi = (\xi_1, \ldots, \xi_n)^T \) is the normalized left eigenvector corresponding to the unique zero eigenvalue of \( \mathcal{L} \). Moreover, we denote with \( L \) the matrix \( -\mathcal{L} \).

III. ADAPTIVE SYNCHRONIZATION OF \( N \) IDENTICAL INTERCONNECTED SYSTEMS

Let us consider a network of \( N \) identical systems. The dynamic of each node is described by \( f(\cdot) \). If we choose the vertex-based strategy, using the definitions above the governing equations of network (3) can be rewritten as follows:

$$ \dot{X} = F(X) + (\sigma \odot L)X, \quad X(0) = X_0 \quad (8) $$

$$ \dot{\sigma} = \mu |LX|, \quad \sigma(0) = \sigma_0 \quad (9) $$

where \( \sigma = 1_n \odot \sigma \), with \( 1_n = [1, \ldots, 1]^T \) being a \( n \)-dimensional vector, \( X = (x_1^T, \ldots, x_N^T)^T \), \( F(X) = (f(x_1)^T, \ldots, f(x_N)^T)^T \) and \( L = L \odot 1_n \) and where we omit the explicit dependence on \( t \).

Furthermore, say \( \Delta = L_n \odot \Delta, U = U \odot 1_n \).

Note that, obviously, if we set \( e(t) = X^T UX \), then \( \lim_{t \to \infty} e(t) = 0 \) iff (2) is verified.

Theorem 1: If \( f \) is QUAD and the matrix [\( \Delta + U(\sigma \odot L) \)] is negative semidefinite for all \( t \geq 0 \), then the synchronization manifold is globally asymptotically stable (G.A.S.)

Proof:

Let us introduce the following candidate Lyapunov function:

$$ V(X, \sigma) = \frac{1}{2} \eta X^T(t)UX(t) + \frac{1}{2\mu} (c - \sigma(t))^T(c - \sigma(t)), \quad (10) $$

where \( \eta \) is a positive scalar and \( c \) is a \( nN \)-dimensional arbitrary vector. We have:

$$ \dot{V} = \eta X^T U[f(X) + (\sigma \odot L)X] - (c - \sigma)^T(LX) = $$

$$ = \eta X^T U[f(X) - \Delta X] + \eta X^T U\Delta X + $$

$$ + \eta X^T U(\sigma \odot L)X - (c - \sigma)^T(LX). \quad (11) $$

Since \( f \) is QUAD, we can state that \( X^T U[f(X) - \Delta X] \leq -\varpi X^T UX \).

Consequently, we can write:

$$ \dot{V} \leq -\varpi X^T UX + \eta X^T U[\Delta + U(\sigma \odot L)]X - (c - \sigma)^T(LX). \quad (12) $$

Let us denote as \( W(X, \sigma) \) the right-hand side of inequality (12) and label as \( W_1(X, \sigma) \), \( W_2(X, \sigma) \), and \( W_3(X, \sigma) \) the first, second and third addend of \( W(X, \sigma) \) respectively.

Clearly, \( W_1(X, \sigma) \) and \( W_2(X, \sigma) \) are negative semi-definite from the assumptions. It suffices, therefore, to show that the term \( -(c - \sigma)^T(LX) \) is also negative semidefinite.

Obviously, if \( \sigma \) is upper bounded, then there exists a value of the arbitrary constant vector \( c \) that guarantees asymptotic stability of system (8)-(9).

Otherwise, if \( \sigma \) were unbounded, as explained below, we would get a contradiction. Indeed, both \( W_2(X, \sigma) \) and \( W_3(X, \sigma) \) are linear functions of \( \sigma \). Hence, if \( \sigma \) diverged, both terms would also diverge linearly. Thus, it is possible to find a suitable value of the constant \( \eta \) so that, for all \( X \) and \( \sigma \), \( |W_2(X, \sigma)| \geq |W_3(X, \sigma)| \). As \( W_2(X, \sigma) \) is negative semi-definite from the hypothesis, we would then get that for all \( X \) and \( \sigma \), \( \dot{V} \leq 0 \) against the assumption that \( \sigma \) diverged. Hence, the various \( \sigma \) are bounded and \( \dot{V} \leq 0 \) for all \( X \in \mathbb{R}^{nN}, \sigma \in \mathbb{R}^n \). As a consequence, also the synchronization error \( e = X^T UX \) is bounded.

Now, to prove that the error approaches zero as \( t \to \infty \), we have to verify that \( \dot{V} \) is bounded (see [9], pp. 199-208). Firstly, note that:

$$ \frac{\partial |LX|}{\partial X} = L, $$

where \( L = [1_n, \ldots, 1_n] \) and \( l_n \) is a \( nN \)-dimensional row-vector defined as follows:

$$ L_n = \begin{cases} L_i & \text{if } L_i X \geq 0 \\ -L_i & \text{if } L_i X < 0 \end{cases}. $$
Moreover, we have:
\[
\frac{\partial [X^T (\sigma \odot L) X]}{\partial \sigma} = A(X) = 
\]
\[
[X_1 \sum_{j=1}^{n} x_j L_{1j}, \ldots, X_{nN} \sum_{j=1}^{n} x_j L_{nN,j}].
\]
Similarly, we can write:
\[
\frac{\partial [X^T UF(X)\] }{\partial X} = \sum_{j=1}^{n} (F_j(X)X_1 + F_j(X))U_{1j}, \ldots,
\]
\[
\sum_{j=1}^{n} (F_j(X)X_n + F_j(X))U_{nj}, \ldots, \sum_{j=1}^{n} (F_j(X)X_nN + + F_j(X))U_{nN,j}] = B(X).
\]
Now we can easily calculate \( \tilde{V} \) as:
\[
\tilde{V} = \frac{\partial V}{\partial X} X + \frac{\partial \tilde{V}}{\partial \sigma} \sigma =
\]
\[
\{B(X) + 2X^T(\sigma \odot L) - (c - \sigma)^T L\} \cdot [F(X) + (\sigma \odot L)X] + (A(X) + [LX]^T)^T) \mu |LX|.
\]
As \( \sigma \) and \( c \) bounded, \( \tilde{V} \) is also bounded and therefore \( \tilde{V} \) is uniformly continuous. So, as shown in [9] (pp. 199-208), we can state that the error goes to zero, i.e. the synchronization manifold is G.A.S.

### IV. NUMERICAL VALIDATION

The next step is now to validate the proposed strategy on a set of representative examples.

#### A. Network of 2 identical Kuramoto oscillators

Let’s consider a simple network of two Kuramoto oscillators [10] connected with the above introduced adaptive strategy. Each Kuramoto is assumed to evolve along the flow generated by \( \dot{\theta}_i = \omega_i \). Choosing \( \omega_1 = \omega_2 = 0 \), the equations of the network are:
\[
\dot{\theta}_1(t) = \sigma(t)(\theta_2(t) - \theta_1(t)), \quad \dot{\theta}_2(t) = \sigma(t)(\theta_1(t) - \theta_2(t)), \quad \dot{\sigma} = \mu |\theta_2(t) - \theta_1(t)|,
\]
and according to our notation, we have:
\[
L = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = 0,
\]
\[
\xi = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \Rightarrow \Xi = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \Rightarrow 
\]
\[
U = \Xi - \xi^T = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}.
\]
In this case we have \( f = 0 \), so we can obviously state that \( f \) is QUAD with \( \Lambda = 0 \). Moreover, it can be trivially shown that \( U(\sigma \odot L) \). Thus, the hypothesis of theorem 1 are verified and so the synchronization manifold \( x_1(t) = x_2(t) \) is G.A.S.

In our simulation, w.l.o.g., we choose \( \mu = 0.1 \) and \( \sigma(0) = 0 \), so that at the beginning the two systems are uncoupled. Moreover, we choose the initial conditions of the two oscillators randomly (in our case \( \theta_1(0) = -1.146 \) and \( \theta_2(0) = 1.1909 \)). According with our theory, in the numerical simulations the synchronous state is asymptotically reached and \( \sigma \) settles to a constant value (see Figs. 1 and 2).

#### B. Networks of \( N \) identical Kuramoto oscillators

We move now to a network of \( N \) identical (i.e. \( \omega_i = \tilde{\omega}, \forall i \)) Kuramoto oscillators and we suppose that, w.l.o.g., \( \tilde{\omega} = 0 \). The strategy used to synchronize the network is the vertex-based adaptive strategy presented above. We can rewrite the governing equations of the network as follow:
\[
\dot{\theta} = (\sigma \odot L)\theta, \quad \dot{\sigma} = \mu |L\theta|.
\]
In our simulation, we consider a network of \( N = 100 \) nodes connected through a Barabasi-Albert scale-free network, with \( m_0 = 5 \) starting nodes (for further details on these topology, see [1], [2], [11]). We choose, as above, \( \mu = 0.1 \) and \( \sigma_i(0) = 0 \) for all \( i = 1, 2, \ldots, N \). Moreover, the initial condition \( \theta_i(0) \) of the oscillators are chosen randomly from a standard normal distribution in the range \([-2.5, 2.5]\).

During the simulations, we have calculated \( U(\sigma \odot L) \) and we have checked numerically that it is always negative semi-definite. As a consequence, the hypothesis of theorem 1 is verified and so the synchronization manifold is G.A.S.
According with this result, Figs. 3 and 4 show that synchronization is achieved and all the $\sigma_i$ settle indeed to a constant value.

V. CONCLUSION

We have shown a possible strategy to prove global asymptotic stability of a novel strategy to synchronize adaptively a network of nonlinear oscillators. The theoretical derivation was illustrated by means of two representative examples. Ongoing work is devoted to simplify the assumptions of Theorem 1, which can be difficult to check in certain circumstances. We wish to emphasize that the approach presented can be successfully used to explain the behaviour of the adaptive strategies presented recently in [7], [8]. The scheme presented can be useful in all those applications where synchronization needs to be attained in the presence of uncertainties and noise, when fixed coupling gains could not guarantee global asymptotic stability.

REFERENCES