State-feedback control of complex dynamical networks

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Abstract—The control issue of complex dynamical networks is considered in this paper, in the case where a control action is locally applied to all the network dynamical nodes. By making use of classical state-feedback control techniques, a complex dynamical network is stabilized to one of its equilibrium points. Both theoretical analysis and numerical simulations are reported in this paper.

1. Introduction

Complex dynamical networks are composed of a large number of dynamical nodes, which may be coupled to each other through diffusion, convection, conduction, etc. [1]. Typical examples include cellular neural networks [2], coupled map lattices [3], arrays of Josephon junctions [4], and coupled chemical reactors [5]. Obviously, the “global” behaviors of such complex networks are determined by both the dynamics at every node and the interactions among them (for instance, electrical activity in neural or cardiac cells is the result of a large number of ionic currents and signaling events [6]).

The complicated dynamical behaviors of complex dynamical networks may result harmful in many cases, therefore controlling the dynamics taking place over complex networks is important. As a matter of fact, the issue of controlling complex dynamical networks has become a focal topic in the control community in the past decade. However, most of the existing works on controlling complex dynamical networks have been concentrated on networks with completely regular topological structures, such as chains, grids, lattices, and fully-connected graphs. The main benefit in using these simple architectures is that it allows for focusing on the complexity caused by the nonlinear dynamics at the nodes without considering additional complexity in the network structure itself. The effects of the network topology on its dynamics, however, cannot be ignored in many cases, as well as its rapid structural evolution over time. For example, it was found [7] that the multicast resource reservation styles in the Internet are quite different from those with the simple linear, tree, or star-shaped topologies. It was also reported [8] that the synchronizability of scale-free networks, characterized by a power-law degree distribution, is strongly affected by the exponent of the power-law.

In this paper, the issue of controlling complex dynamical networks is considered. Firstly, a linearly coupled dynamical network is introduced. Then, based on the conventional state-feedback control method, a sufficient condition on controlling a complex dynamical network to its equilibrium point is obtained. Finally, numerical simulations are given to verify the effectiveness of the given method.

2. Problem description

Suppose that a complex network consists of \( N \) linearly and diffusively coupled identical nodes, with each node being an \( n \)–dimensional dynamical system. The state equations of this dynamical network are given by

\[
\dot{x}_i = f(x_i) + c \sum_{j=1}^{\gamma_{ij}} a_{ij} \Gamma x_j, \quad i = 1,2,\ldots, N. \tag{1}
\]

Here, \( x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n \) are the state variables of node \( i \), the constant \( c > 0 \) represents the coupling strength, \( \Gamma = (\gamma_{ij}) \in \mathbb{R}^{n \times n} \) is a symmetric matrix linking coupled variables, and if for some pairs \((i,j)\), \(1 \leq i, j \leq n\), \( \gamma_{ij} \neq 0 \), it means two coupled nodes are linked through their \( i \)-th and \( j \)-th state variables, respectively. Here, we assume that each pair of coupled oscillators is linked through the complete set of identical sub-state variables, i.e., \( \Gamma = diag(1, \ldots, 1) \).

In network (1), the symmetric coupling matrix \( A = (a_{ij}) \in \mathbb{R}^{N \times N} \) represents the connectivity configuration of an undirected complex network. If there is a connection between node \( i \) and node \( j \) (\( i \neq j \)), then \( a_{ij} = a_{ji} = 1 \); otherwise, \( a_{ij} = a_{ji} = 0 \) (\( i \neq j \)).

The degree \( k_i \) of node \( i \) is defined to be the number of connections incident in it:

\[
\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji} = k_i, \quad i = 1,2,\ldots, N. \tag{2}
\]

Let the diagonal elements be \( a_{ii} = -k_i, \quad i = 1,2,\ldots, N \). Suppose the network is connected in the sense of having
no isolated clusters, i.e., the symmetric matrix $A$ is irreducible. We know that zero is an eigenvalue of $A$ with multiplicity 1, and other eigenvalues of $A$ are strictly negative in the order: $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$.

The objective here is to stabilize network (1) onto an equilibrium point of the network, in the sense that

$$x_i(t) = x_2(t) = \cdots = x_N(t) \to \bar{x}, \text{ as } t \to \infty,$$  \hspace{1cm} (3)

where $\bar{x} \in \mathbb{R}^n$ is a solution of an isolated node, satisfying $\dot{\bar{x}} = f(\bar{x})$.

Without loss of generality, we assume that network (1) satisfies the following assumption:

**Assumption:** For any isolated node, there exists a non-negative constant $\beta$ such that

$$(x - y)^T (f(x) - f(y)) \leq \beta (x - y)^T (x - y),$$  \hspace{1cm} (4)

where $\beta > 0$ is a constant.

### 3. Main result

To achieve the goal (3), the state feedback control strategy is applied to stabilize network (1). Therefore, the controlled network (1) can be described by

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} a_{ij} x_j + u_i, \quad i = 1, 2, \cdots, N.$$  \hspace{1cm} (5)

Here $u_i$ is the controller of node $i$, described by

$$u_i = -k(x_i - \bar{x}), \quad i = 1, 2, \cdots, N,$$  \hspace{1cm} (6)

where $k > 0$ is the constant control strength.

Define $e_i(t) = x_i(t) - \bar{x}$; then one can have

$$\dot{e}_i = f(x_i) - f(\bar{x}) + c \sum_{j=1}^{N} a_{ij} e_j - k e_i, \quad i = 1, \cdots, N.$$  \hspace{1cm} (7)

If the given assumption holds, we can get a sufficient condition for stabilizing network (1):

**Theorem:** Assume the given assumption holds. Network (5) is globally stabilized to $\bar{x}$ if

$$\lambda_{\max}((\beta-k)I + c A) < 0,$$  \hspace{1cm} (8)

where $\lambda_{\max}(M)$ is the highest eigenvalue of matrix $M$.

**Proof:** Construct a Lyapunov functional of the form

$$V = \frac{1}{2} \sum_{i=1}^{N} e_i^T e_i$$  \hspace{1cm} (9)

The time derivative of (9) along trajectories of (7) is

$$\dot{V} = \sum_{i=1}^{N} e_i^T (f(x_i) - f(\bar{x})) + \sum_{i=1}^{N} e_i^T \left( c \sum_{j=1}^{N} a_{ij} x_j \right) - k \sum_{i=1}^{N} e_i^T e_i$$

Define

$$\tilde{e}_i = \begin{bmatrix} e_{1j} \\ e_{2j} \\ \vdots \\ e_{Nj} \end{bmatrix}, \quad j = 1, 2, \cdots, n,$$

one has

$$\dot{V} \leq \sum_{j=1}^{N} \beta \tilde{e}_j^T e_j + \sum_{j=1}^{N} \left( e_j^T \left( c \sum_{j=1}^{N} a_{ij} x_j \right) \right) - k \sum_{i=1}^{N} e_i^T e_i$$

$$= \beta \sum_{j=1}^{N} \tilde{e}_j^T e_j + c \sum_{j=1}^{N} \tilde{e}_j^T A \tilde{e}_j - k \sum_{i=1}^{N} e_i^T e_i$$

$$= \sum_{j=1}^{N} \tilde{e}_j^T ((\beta - k)I + c A) \tilde{e}_j$$

Obviously, $\dot{V} < 0$ if $\lambda_{\max}((\beta-k)I + c A) < 0$. \hspace{1cm} (10)

This completes the proof of the theorem.

**Corollary:** Assume the given assumption holds. Network (5) is globally stabilized to $\bar{x}$ if $k > \beta$.

**Proof:** Define $\tilde{e}_j = P v_j$ ($j = 1, 2, \cdots, n$) where matrix $P$ is such that $P^T A P = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$, one has

$$\tilde{e}_j^T ((\beta - k)I + c A) \tilde{e}_j$$

$$= v_j^T ((\beta - k)I + c \times \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)) v_j$$

$$= \sum_{j=1}^{N} ((\beta - k + c \lambda_j) v_j^2$$  \hspace{1cm} (j = 1, 2, \cdots, n)

Clearly, $\dot{V} < 0$ if $k > \beta$.

This completes the proof of the corollary.

### 4. Numerical Simulations

Consider a chaotic dynamical network of Lorenz systems, where a single Lorenz oscillator is described by

$$\begin{align*}
\dot{x}_1 &= f(x) = \begin{bmatrix} p_1 (x_2 - x_1) \\ p_3 x_1 - x_1 x_3 - x_2 \\ x_1 x_2 - p_2 x_3 \end{bmatrix} \\
\dot{x}_2 &= f(x) = \begin{bmatrix} p_1 (x_2 - x_1) \\ p_3 x_1 - x_1 x_3 - x_2 \\ x_1 x_2 - p_2 x_3 \end{bmatrix} \\
\dot{x}_3 &= f(x) = \begin{bmatrix} p_1 (x_2 - x_1) \\ p_3 x_1 - x_1 x_3 - x_2 \\ x_1 x_2 - p_2 x_3 \end{bmatrix}
\end{align*}$$  \hspace{1cm} (12)

with the parameter set of $p_1 = 10$, $p_2 = \frac{8}{3}$, $p_3 = 28$, and $\beta = 60$ in the assumption.

We generate a scale-free network with the BA model [9], which has 200 nodes with parameters $m = m_0 = 2$, and set the coupling strength $c = 0.01$. To stabilize such a BA network to the homogenous state $x = (0, 0, 0)$, we set the control strength $k = 62.5$. Since
\[ \lambda_{\text{max}}((\beta - k)I + cA) = -2.5 < 0, \] the condition in the theorem is satisfied.

States \( x \) and \( y \) of the highest degree node in the network are shown in Figs. 1-2. From 0 to 40 seconds, we do not have any control to nodes. Consequently, chaotic phenomenon is observed. After 40 seconds, all nodes are controlled, and then they tend to \( \mathbf{x} = (0,0,0) \) quickly.

Similarly, we construct a small-world network with the NW small-world model [10] with network parameters \( N = 200, K = 2 \) and \( p = 0.01 \). We set the coupling strength \( c = 0.01 \) and the control strength \( k = 63 \). From Figs. 3-4, one can see that the network is well stabilized to its equilibrium point \( \mathbf{x} = (0,0,0) \).

5. Conclusion

The stabilization of complex dynamical networks onto a homogenous state has been studied in this paper. Based on a classical state-feedback control strategy, a sufficient condition was obtained to stabilize a dynamical network with a fixed coupling strength. Numerical simulations have shown that the proposed method can be used to stabilize such networks as BA scale-free and NW small-world dynamical networks. However, a real network is usually very large, which makes almost impossible to control all the nodes of the network. Therefore, the pinning control strategy [11-12], in which only a portion of nodes in the network is controlled, may benefit the control of many real-world dynamical networks. Another important aspect is represented by the role played by the network topology on the effectiveness of the control action (in an analogous way to the propensity of complex networks to being synchronized, depending on some of their structural properties [13-14]). These topics will be discussed in our future research.
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References